B.S.T.J. BRIEFS

Realizability Conditions for the Impedance Function of the Lossless Tapered Transmission Line— A Critique

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I. INTRODUCTION

In a recent brief[1] in the B.S.T.J., Zador presents, without proof, realizability conditions for the input impedance of the lossless tapered transmission line terminated in unit resistance. Upon a careful examination of the brief, it appears that the conditions are not accurate. The following analysis clarifies this point and, incidentally, provides alternatives to Zador's necessary conditions.

Consider a nonuniform line (Fig. 1) with inductance per unit length $\mathcal{L}(x)$ and capacitance per unit length $\mathcal{C}(x)$ such that (to follow Zador)

$$\mathcal{L}(x)\mathcal{C}(x) = 1.$$

Let $V(x,s)$ and $I(x,s)$ be the voltage and current along the line with polarities as indicated in Fig. 1. The equations of the line are

$$\frac{dV(x,s)}{dx} = -s\mathcal{L}(x)I(x,s)$$

$$\frac{dI(x,s)}{dx} = -s\mathcal{C}(x)V(x,s).$$

Eliminating $I(x,s)$ and taking into account that $\mathcal{L}(x) = 1/\mathcal{C}(x)$ we get

$$\frac{d}{dx}\left(\mathcal{C}(x)\frac{dV(x,s)}{dx}\right) = s^2\mathcal{C}(x)V(x,s).$$

Note also that

$$I(x,s) = -\frac{\mathcal{C}(x)}{s}\frac{dV(x,s)}{dx}.$$

Hence, we can identify Zador's $y(x,s)$ and $c(x)$ with $V(x,s)$ and $\mathcal{C}(x)$, respectively. From the reference polarities of the voltages and currents in Fig. 1, we see that for a unit resistance termination at $x = 0$ we must

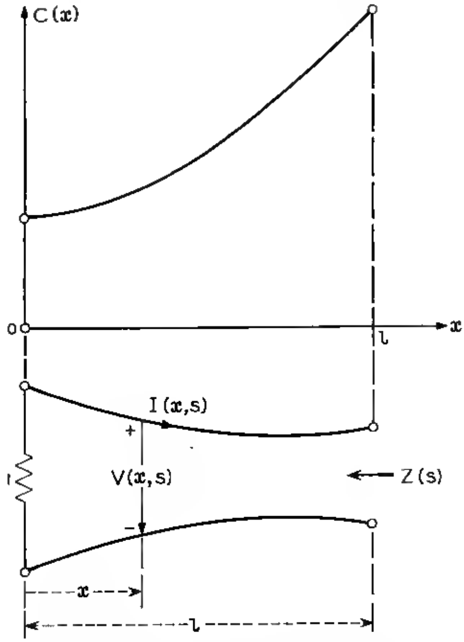

Fig. 1.—Lossless tapered transmission line.

have

$$V(0,s) = -I(0,s).$$

Hence, if we impose the condition (following Zador)

$$y(0,s) = V(0,s) = -i$$

then for unit resistance termination we should have

$$y'(0,s) = \frac{dV(0,s)}{dx} = \frac{sV(0,s)}{c(0)} = -\frac{is}{c(0)}.$$

The driving point impedance, for any termination, should read

$$Z(s) = \frac{s}{c(l)} \frac{y(l,s)}{y'(l,s)}.$$

Thus, the signs are wrong in Ref. 1. This is not the crucial error however.

In this paper, we will show that the difficulties in Zador's paper arise from the following facts:

(*i*) He does not consider the matched line. Unmatched lines tend to have almost periodic behavior for large real frequencies and hence

the network functions do not have limits at infinity. This point will be made more precise in the sequel.

(*ii*) Multiplication of $Z(j\omega)$ hy exp $(-2jl\omega)$ in property (*iii*) of the necessity statement introduces periodic hehavior at infinity *even in the matched case.*

(*iii*) Physical meaning has not heen attached to the N_i and D_i. These should ohviously he identified with the well-known ABCD parameters to correct (*ii*) of the necessity conditions.

II. COMMENTS ON ZADOR'S BRIEF

Property (*iii*) in the necessity statement does not appear to he true as stated. One can easily construct many counter examples.

Example 1: The uniform line with (following Zador's notation) $c(x) = 1$ and length $l = 1$, terminated in a 1-ohm resistor. Ohviously $c(x)$ satisfies the conditions stipulated hy Zador, i.e., $c(x)$ is positive and continuously differentiahle in the interval $0 \leqq x \leqq 1$. Clearly the driving point impedance is

$$Z(j\omega) = 1.$$

Therefore,

$$f(\omega) = \operatorname{Re} \exp(-2jl\omega)Z(j\omega) = \cos 2\omega.$$

Clearly $\cos 2\omega$ does not have a limit for $\omega \to \pm\infty$.

Consider now a less trivial counter example.

Example 2: The exponential line terminated in a unit resistance. With Zador's notation $c(x) = \exp 2x$, and $l = 1$. In this case hy solving Zador's (1) with the suhsequent houndary conditions (appropriately corrected) we find

$$Z(j\omega) = \frac{A(j\omega) + B(j\omega)}{C(j\omega) + D(j\omega)}, \tag{1}$$

where

$$A(j\omega) = \frac{1}{e}\left\{\cos\sqrt{\omega^2 - 1} + \frac{\sin\sqrt{\omega^2 - 1}}{\sqrt{\omega^2 - 1}}\right\} \tag{2}$$

$$B(j\omega) = \frac{1}{e}\left\{j\omega\,\frac{\sin\sqrt{\omega^2 - 1}}{\sqrt{\omega^2 - 1}}\right\} \tag{3}$$

$$C(j\omega) = e\left\{j\omega\,\frac{\sin\sqrt{\omega^2 - 1}}{\sqrt{\omega^2 - 1}}\right\} \tag{4}$$

$$D(j\omega) = e\left\{\cos \sqrt{\omega^2 - 1} - \frac{\sin \sqrt{\omega^2 - 1}}{\sqrt{\omega^2 - 1}}\right\}. \tag{5}$$

It turns out that

$$R = \operatorname{Re} Z(j\omega) = \frac{1}{D^2 - C^2} = \frac{1}{K(\omega)} \tag{6}$$

$$X = \operatorname{Im} Z(j\omega) = -j\frac{BD - AC}{D^2 - C^2} = -\frac{2\omega \sin^2 \sqrt{\omega^2 - 1}}{K(\omega)(\omega^2 - 1)}, \tag{7}$$

where

$$\frac{K(\omega)}{e^2} = \left\{\cos \sqrt{\omega^2 - 1} - \frac{\sin \sqrt{\omega^2 - 1}}{\sqrt{\omega^2 - 1}}\right\}^2 + \frac{\omega^2 \sin^2 \sqrt{\omega^2 - 1}}{\omega^2 - 1}. \tag{8}$$

Hence,

$$\begin{aligned} f(\omega) &= \operatorname{Re} \exp(-2j\omega)Z(j\omega) = R \cos 2\omega + X \sin 2\omega \\ &= \frac{1}{K(\omega)}\left[\cos 2\omega - \frac{2\omega \sin^2 \sqrt{\omega^2 - 1} \sin 2\omega}{\omega^2 - 1}\right]. \end{aligned} \tag{9}$$

Obviously $f(\omega)$ does not possess a limit for $\omega \to \pm\infty$.

Example 3: Consider now the class of transmission lines which have a positive bounded and twice differentiable $c(x)$ in the interval $0 \leqq x \leqq l$. It can be shown (see e.g., Ref. 2) that the ABCD parameters satisfy the following asymptotic relations, for ω large:*

$$A(j\omega) = \sqrt{\frac{c(0)}{c(l)}} \cos l\omega + O\left(\frac{1}{\omega}\right) \tag{10}$$

$$B(j\omega) = j\frac{\sin l\omega}{\sqrt{c(0)c(l)}} + O\left(\frac{1}{\omega}\right) \tag{11}$$

$$C(j\omega) = j\sqrt{c(0)c(l)} \sin l\omega + O\left(\frac{1}{\omega}\right) \tag{12}$$

$$D(j\omega) = \sqrt{\frac{c(l)}{c(0)}} \cos l\omega + O\left(\frac{1}{\omega}\right). \tag{13}$$

These results follow from the classical theory of the asymptotic behavior of the eigenfunctions of Sturm-Liouville problems.[3] The WKBJ method is a related subject. Schelkunoff has discussed these

* The line is driven at the point $x = l$. The product of the inductance per unit length and the capacitance per unit length is assumed to be unity.

matters in an elementary way in at least one of his textbooks[4] (he does not include the $O(1/\omega)$ term).

If the line is terminated at $x = 0$ with a resistance R_0, we have for the driving point impedance

$$Z(j\omega) = \frac{R_0 A(j\omega) + B(j\omega)}{R_0 C(j\omega) + D(j\omega)}. \tag{14}$$

Substituting from (10), (11), (12), and (13) we find that for large ω

$$Z(j\omega) = \frac{R_0 c(0)}{c(l)} \left[1 + j \frac{(1 - R_0^2 c^2(0)) \sin l\omega}{R_0 c(0) \{ \cos l\omega + j R_0 c(0) \sin l\omega \}} + O\left(\frac{1}{\omega}\right) \right] \tag{15}$$

and

$$R = \operatorname{Re} Z(j\omega) = \frac{R_0 c(0)}{c(l)} \left[1 + \frac{(1 - R_0^2 c^2(0)) \sin^2 l\omega}{1 - (1 - R_0^2 c^2(0)) \sin^2 l\omega} + O\left(\frac{1}{\omega}\right) \right] \tag{16}$$

$$X = \operatorname{Im} Z(j\omega) = \frac{(1 - R_0^2 c^2(0)) \sin 2l\omega}{2c(l)[1 - (1 - R_0^2 c^2(0)) \sin^2 l\omega]} + O\left(\frac{1}{\omega}\right). \tag{17}$$

Hence, if $R_0 c(0) \neq 1$, $Z(j\omega)$, $\operatorname{Re} Z(j\omega)$, and $\operatorname{Im} Z(j\omega)$ do not have limits for $\omega \to \pm\infty$.

Similarly, $f(\omega) = \operatorname{Re} \exp(-2jl\omega) Z(j\omega)$ does not have a limit for $\omega \to \pm\infty$. When $R_0 c(0) = 1$, i.e., when the line is "locally matched" at $x = 0$, we have

$$Z(j\omega) = \frac{1}{c(l)} + O\left(\frac{1}{\omega}\right) \tag{18}$$

$$R = \operatorname{Re} Z(j\omega) = \frac{1}{c(l)} + O\left(\frac{1}{\omega}\right) \tag{19}$$

$$X = \operatorname{Im} Z(j\omega) = O\left(\frac{1}{\omega}\right). \tag{20}$$

In this case,

$$f(\omega) = \operatorname{Re} \exp(-2jl\omega) Z(j\omega) = \frac{1}{c(l)} \cos 2l\omega + O\left(\frac{1}{\omega}\right). \tag{21}$$

Clearly, $f(\omega)$ does not have the asymptotic behavior stipulated by Zador; it does not even have a limit (because of the $\cos 2l\omega$ term).

Note that the asymptotic formulas (10), (11), (12), and (13) are also valid for a continuous positive $c(x)$ which is *piecewise* twice differentiable. This can be proven by partitioning the line at the discontinuity points and finding the overall ABCD matrix by multiplying

the ABCD matrices of the sections of the line which now have a twice differentiahle $c(x)$.

Hence, property (*iii*) of Zador's necessity statement could he replaced by the following: If (*i*) $c(x)$ is a positive continuous and piecewise twice differentiahle function of the real variahle x, (*ii*) the line is terminated in a unit resistance and $c(0) = 1$, then the following relation is valid for large ω:

$$Z(j\omega) = \frac{1}{c(l)} + O\left(\frac{1}{\omega}\right). \tag{22}$$

Another suhstitute will he discussed in the following. Let $\rho(j\omega)$ be the voltage reflection coefficient at $x = l$ for the unit resistance terminated line, then

$$Z(j\omega) = \frac{1}{c(l)} \frac{1 + \rho(j\omega)}{1 - \rho(j\omega)}. \tag{23}$$

For a $c(x)$ which is continuous and twice differentiable in the interval $0 \leqq x \leqq l$ with

$$\begin{aligned} c(0) &= 1 \\ \frac{dc(0)}{dx} &= \frac{dc(l)}{dx} = 0 \end{aligned} \tag{24}$$

we can see, using Schelkunoff's results on wave propagation in stratified media,[5] that for ω large

$$\rho(j\omega) = O\left(\frac{1}{\omega^2}\right). \tag{25}$$

From (23) we have in general for $| \rho(j\omega) | < 1$

$$Z(j\omega) = \frac{1}{c(l)} \{1 + 2\rho(j\omega) + 2\rho^2(j\omega) + \cdots\}. \tag{26}$$

Hence, using (25) we get

$$Z(j\omega) = \frac{1}{c(l)} + O\left(\frac{1}{\omega^2}\right) \tag{27}$$

for large ω.

To generalize (following Schelkunoff[5]) if $c(0) = 1$ and the first n derivatives of $c(x)$ are continuous functions of x and vanish at the boundaries then for large ω

$$\rho(j\omega) = O\left(\frac{1}{\omega^{n+1}}\right) \tag{28}$$

and therefore,

$$Z(j\omega) = \frac{1}{c(l)} + O\left(\frac{1}{\omega^{n+1}}\right). \tag{29}$$

Property (*ii*) in the necessity statement of Zador is also wrong.

Proof: The input impedance of the unit-resistance terminated line may be written, in terms of the ABCD parameters, as follows:

$$Z(s) = \frac{A(s) + B(s)}{C(s) + D(s)} = \frac{Q(s)}{P(s)}. \tag{30}$$

Consider a line with a twice differentiable $c(x)$. In this case $A(s)$, $B(s)$, $C(s)$, and $D(s)$ are entire functions of order 1 and type l (see Ref. 2), i.e.,

$$\begin{aligned} A(s) &\approx c_1 e^{ls} \\ B(s) &\approx c_2 e^{ls} \\ C(s) &\approx c_3 e^{ls} \\ D(s) &\approx c_4 e^{ls} \end{aligned} \tag{31}$$

(where c_1, c_2, c_3, c_4 are positive constants) for real $s \to +\infty$. Note also that

$$\begin{aligned} A(s) &= A(-s) \qquad & D(s) &= D(-s) \\ B(s) &= -B(-s) \qquad & C(s) &= -C(-s) \end{aligned} \tag{32}$$

and

$$AB - CD = 1. \tag{33}$$

In order to find Zador's representation with the N_i, D_i ($i = 1,2$) functions we should be able to find an entire function $\varphi(s) \not\equiv 0$ such that when we multiply both the numerator and denominator of $Z(s)$ in (30) by this entire function, we get functions N_i, D_i ($i = 1,2$) with the properties stipulated by Zador.

We will have

$$N_1(s) = \text{Ev}\,[Q(s)\varphi(s)] = A(s)\,\frac{\varphi(s) + \varphi(-s)}{2} + B(s)\,\frac{\varphi(s) - \varphi(-s)}{2} \tag{34}$$

$$N_2(s) = \text{Odd}\,[Q(s)\varphi(s)] = A(s)\,\frac{\varphi(s) - \varphi(-s)}{2} + B(s)\,\frac{\varphi(s) + \varphi(-s)}{2}. \tag{35}$$

Similarly,

$$D_1(s) = D(s)\frac{\varphi(s) + \varphi(-s)}{2} + C(s)\frac{\varphi(s) - \varphi(-s)}{2} \tag{36}$$

$$D_2(s) = D(s)\frac{\varphi(s) - \varphi(-s)}{2} + C(s)\frac{\varphi(s) + \varphi(-s)}{2}. \tag{37}$$

Hence,

$$N_1(s)D_1(s) - N_2(s)D_2(s) = \varphi(s)\varphi(-s). \tag{38}$$

From (34), (35), (36), and (37) it follows that the functions $(\varphi(s) + \varphi(-s))/2$ and $(\varphi(s) - \varphi(-s))/2$ should be of type 0 in order that Zador's N_i and D_i be of type l. Consequently, the functions $\varphi(s)$ and $\varphi(-s)$ themselves are of type 0. Therefore, it is impossible to find an $\varphi(s)$ such that $\varphi(s)\varphi(-s) = \exp 2ls$ as Zador stipulates. So property *(ii)* in Zador's necessity statement could be replaced by

$$N_1D_1 - N_2D_2 = k^2,$$

where k is a constant. Then N_i, D_i $(i = 1,2)$ are proportional to the ABCD parameters with proportionality factor k.

From the above it follows that the sufficiency part as stated is inaccurate. It might be possible to alter the sufficiency conditions to make them valid. In this case a proof must be given. The author has done related work[6] on realizability conditions for nonuniform RC lines and is familiar with the difficulties involved in proving sufficiency conditions of this form.

Finally, Zador's conjectures do not have an obvious physical interpretation and hence they should be justified.

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